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Spectral flow and level spacing of edge states for quantum Hall Hamiltonians

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Abstract

We consider a non-relativistic particle on the surface of a semi-infinite cylinder of circumference L submitted to a perpendicular magnetic field of strength B and to the potential of impurities of maximal amplitude w. This model is of importance in the context of the integer quantum Hall effect. In the regime of strong magnetic field or weak disorder $B \gg w$, it is known that there are chiral edge states, which are localized within a few magnetic lengths close to, and extended along the boundary of the cylinder, and whose energy levels lie in the gaps of the bulk system. These energy levels have a spectral flow, uniform in L, as a function of a magnetic flux which threads the cylinder along its axis. Through a detailed study of this spectral flow, we prove that the spacing between two consecutive levels of edge states is bounded below by $2\pi\alpha L^{-1}$ with $\alpha > 0$, independent of L, and of the configuration of impurities. This implies that the level repulsion of the chiral edge states is much stronger than that of extended states in the usual Anderson model and their statistics cannot obey one of the Gaussian ensembles. Our analysis uses the notion of relative index between two projections and indicates that the level repulsion is connected to topological aspects of quantum Hall systems.

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1. Introduction and results

Recently there has been mathematical progress concerning the spectral properties of disordered quantum Hall systems with boundaries. In the theory of the integer quantum Hall effect one considers non-interacting electrons confined on the surface of a finite cylinder [1] or on a corbino disc [2], submitted to a perpendicular uniform magnetic field of strength B and to the potential of impurities of maximal amplitude w. In a classic paper on the subject [2] Halperin argued that, at least for strong magnetic field and weak disorder ($B \gg w$ in appropriate units), there exist quantum mechanical states localized near and extended along the boundaries of the sample. These states carry a diamagnetic current contributing to the total Hall current. Halperin's analysis applies to energies that lie in the gaps separating the Landau bands of the bulk-disordered Hamiltonian, i.e. the Hamiltonian of an infinite two-dimensional planar system (with no boundaries). Here we will call this part of the spectrum the 'pure edge spectrum'. Progress towards the characterization of the nature of the pure edge spectrum has been made in recent works for systems with one smooth boundary [3–5]. In the present contribution we obtain new results for such systems, which are used in separate work on more realistic geometries involving two boundaries [6].

We consider the Hamiltonian of a particle on a cylinder of radius $\frac{L}{2\pi}$ thread by a flux line with flux Φ

$$H(\Phi) = \frac{1}{2}p_x^2 + \frac{1}{2}\left(p_y - Bx + \frac{\Phi}{L}\right)^2 + W(x) + V(x, y) \tag{1.1}$$

where $x \in \mathbb{R}, -\frac{L}{2} \leqslant y \leqslant \frac{L}{2}$, with periodic boundary conditions in the y direction $\Psi\left(x, -\frac{L}{2}\right) = \Psi\left(x, \frac{L}{2}\right)$. The particle is confined to the left half of the cylinder because of the external potential W which models the boundary of a 'semi-infinite cylinder'. We assume that it is continuous, and $W(x) = 0$ for $x \leqslant 0$, $W'(x) > 0$ for $x \geqslant 0$, $W(x) \to +\infty, x \to +\infty$. For technical reasons we assume a growth of W that is not too fast: we suppose that for $x \geqslant 0$, $u_1 x^\gamma \leqslant W(x) \leqslant u_2 x^\gamma$, for some $0 < u_1 < u_2$ and $\gamma \geqslant 2$. The potential of impurities V is piecewise continuous and bounded $|V(x, y)| \leqslant w$ with $0 < w < \frac{B}{2}$. We also suppose that $V(x, y) = 0$ for $x > 0$, however, our methods can be adapted to a more general model where the impurity potential extends inside the region of the boundary.

We will also use two other Hamiltonians: the 'edge Hamiltonian' $H_e(\Phi)$ obtained from (1.1) by removing V and the 'bulk Hamiltonian' $H_b(\Phi)$ obtained from (1.1) by removing W.

The 'semi-infinite planar' case corresponds to $L = +\infty$. In this limit the corresponding Hamiltonians become independent of Φ and we denote them $H_\infty, H_{e,\infty}, H_{b,\infty}$. It is easy to see that $H_{b,\infty}$ has gaps $G_n \supset \left]\left(n + \frac{1}{2}\right)B + w, \left(n + \frac{3}{2}\right)B - w\right[, n \in \mathbb{N}$. A basic fact is that for weak enough disorder the 'pure edge spectrum' $\sigma(H_\infty) \cap G_n, n \in \mathbb{N}$ is continuous. This result is also proved for W replaced by a Dirichlet boundary condition at $x = 0$ and for smooth-curved open boundaries (see [3–5]).

When L is finite G_n contains only discrete isolated eigenvalues. We formulate this result and all the subsequent ones in the special case $n = 0$.

Lemma 1. *Let $B > 2w$. For any $0 < \epsilon < \frac{B}{2} - w$ the set $\sigma(H(\Phi)) \cap \tilde{G}_0, \tilde{G}_0 = \left]\frac{B}{2} + w + \epsilon, \frac{3B}{2} - w - \epsilon\right[$ contains only a finite number of isolated eigenvalues of finite multiplicity. We label the eigenvalues of $H(0)$ in $\tilde{G}_0$ as $E_1(0) \leqslant E_2(0) \leqslant \cdots \leqslant E_N(0)$ for some finite N. Any $E_k(0) \in \tilde{G}_0$ can be continued into one or several analytic branches $E_k(\Phi)$ for $\Phi \in [0, \Phi_k]$ for some small enough $\Phi_k > 0$.*

The discreteness of the spectrum in the specified interval is non-trivial even if the circumference of the cylinder is finite because the impurity potential can extend to infinity in the direction $x \to -\infty$ where there is no confinement. In fact one can see that the rest of the spectrum may have dense parts. For example, if V is a typical realization of a random potential the Landau bands $\left[\left(n + \frac{1}{2}\right)B - w, \left(n + \frac{1}{2}\right)B + w\right]$ have dense spectrum. Now let $0 < \delta < \frac{B}{2} - w - \epsilon$ and $\Delta =]B - \delta, B + \delta[$. For L large enough, as long as an eigenvalue $E_k(\Phi) \in \Delta$ for some Φ, then we are assured that it can be continued into an analytic branch for the whole interval $[0, 2\pi]$. This comes from the fact (see inequality (3.15)) that the maximal variation of $E_k(\Phi)$ is $2\pi\sqrt{3B}L^{-1}$, so that it stays in $\tilde{G}_0$ and never merges in the Landau bands.

In the rest of this work we fix ϵ small and $0 < \delta < \frac{B}{2} - w - \epsilon$, and look only at eigenvalues $E_k(\Phi) \in \Delta$. Note that as Φ varies from 0 to 2π some of the branches may move in or out of Δ. A reformulation of the analysis in [3–5] shows that there exists a spectral flow which is uniform in L. This is expressed by the following lemma.

Lemma 2. *Let $B > 2w$. There exist δ, w_0 small enough, L_0 large enough such that for $w < w_0$, $L > L_0$ all eigenvalues $E_k(\Phi) \in \Delta$ satisfy*

$$L\frac{\mathrm{d}}{\mathrm{d}\Phi}E_k(\Phi) \geqslant \alpha \tag{1.2}$$

where α is strictly positive independent of L and k and depends only on W, B, w and δ.

The existence of a spectral flow is equivalent to the presence of a chiral diamagnetic current. Indeed by the Feynman–Hellman theorem

$$\frac{\mathrm{d}}{\mathrm{d}\Phi}E_k(\Phi) = j_k(\Phi) \tag{1.3}$$

where

$$j_k(\Phi) = \frac{1}{L}\left\langle \Psi_k(\Phi) \middle| \left(p_y - Bx + \frac{\Phi}{L}\right) \Psi_k(\Phi)\right\rangle \tag{1.4}$$

is the diamagnetic current (or edge current) associated with the eigenstate $|\Psi_k(\Phi)\rangle$ corresponding to the level $E_k(\Phi)$.

The Hamiltonians $H(\Phi)$ and $H(\Phi + 2\pi)$ are unitarily equivalent, the unitary operator being multiplied by $\exp\left(2\pi \mathrm{i}\frac{y}{L}\right)$. Thus for each $E_k(\Phi)$ which does not merge in the Landau bands there must exist some k' such that $E_k(2\pi) = E_{k'}(0)$. From lemma 2 it is clear that $k' > k$, but this does not completely characterize the spectral flow. Our main new result states that $k' = k + 1$ and characterizes the level spacing for the pure edge spectrum.

Theorem 1. *Let $B > 2w$. There exist δ, w_0 small enough, L_0 large enough such that for $w < w_0$, $L > L_0$, the branches $E_k(\Phi)$ belonging to Δ for all $\Phi \in [0, 2\pi]$ satisfy*

$$E_k(2\pi) = E_{k+1}(0). \tag{1.5}$$

Moreover the level spacing in Δ satisfies

$$\frac{2\pi\alpha}{L} \leqslant |E_{k+1}(0) - E_k(0)| \leqslant \frac{2\pi\sqrt{3B}}{L}. \tag{1.6}$$

For the constant α in lemma 2 and theorem 1 we can take the right-hand side of (2.29). The important point is that in the lower bound of (1.6) α does not depend on the detailed configuration of the impurity potential but only on its maximal amplitude. So for a random potential the level spacing is random but our lower bound is non-random.

For the usual Anderson model it is proved that the level spacing of localized states satisfies the Poisson statistics [7, 8], and it is numerically established that extended states have a level repulsion satisfying the Wigner surmise [9]. Here we have a different situation: the states are extended, chiral and have a much stronger level repulsion which makes the level spacing very rigid. Let $\rho(E)$ denote the average density of edge states. We expect from (1.6) that, in the limit $L \to \infty$, the rescaled level spacing $s = L\rho(E_k)|E_{k+1} - E_k|$ has a histogram $p(s)$ which is a certain broadening of $\delta(s - 1)$ with a finite support of $O\left(\frac{w^2}{B^2}\right)$. The level statistics cannot follow the Gaussian ensembles and it would be worthwhile to investigate this question numerically for an analogous model on a lattice. It is apparent from the proof of theorem 1 that the rigidity of the edge spectrum is related to the topological invariants of the quantum Hall effect. Also if the spectral flow satisfied $E_k(2\pi) = E_{k+n}(0)$ with $n \geqslant 2$, it would not be forbidden to have n consecutive levels arbitrarily close.

We wish to point out that all these features can be checked immediately for a simple toy Hamiltonian. Consider a one-dimensional chiral particle on a circle of circumference L threaded by a flux Φ

$$h(\Phi) = \left(-\mathrm{i}\partial_y + \frac{\Phi}{L}\right) + v(y). \tag{1.7}$$

The exact spectrum is

$$e_m(\Phi) = \frac{2\pi m}{L} + \frac{\Phi}{L} + \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\, v(y) \tag{1.8}$$

which satisfies (1.2), (1.5), (1.6) and has $p(s) = \delta(s-1)$. It is expected that (1.7) is a good approximation of (1.1) for distances to the boundary of the order of the magnetic length $x = O\big(\frac{1}{\sqrt{B}}\big)$.

Finally we recall how it follows from (1.5) that the 'edge conductance' of the semi-infinite system is quantized (see [1, 2, 4] for similar discussions). Let $P_\Delta(\Phi)$ be the projector of $H(\Phi)$ on an energy range Δ. The edge conductance may be defined as the total edge current per unit energy

$$\sigma_e = \lim_{L\to\infty} \frac{1}{|\Delta| L} \mathrm{Tr}\left(p_y - Bx - \frac{\Phi}{L}\right) P_\Delta(\Phi). \tag{1.9}$$

We assume that for a suitable class of potentials V this limit exists and is independent of Φ (the flux has no effect for the semi-infinite plane). We expect this assumption to be true for typical realizations of random potentials that are ergodic with respect to the translations along y. In this case the limit should be equal to $\frac{1}{\Delta}\mathrm{Av}\int \mathrm{d}x \langle x, 0|(p_y - Bx)P_{\infty,\Delta}|x, 0\rangle$ where Av is the average over the disorder and $P_{\infty,\Delta}$ the projector of H_∞ onto Δ. The limit of the latter quantity when $\Delta \to \mu$ has been shown to be an integer if μ is a point in the gap G_0, by non-commutative geometry techniques applied to the lattice case [10]. In the present situation it is easy to see that for Δ in the first gap of the bulk Hamiltonian $H_{b,\infty}$

$$\begin{aligned}
\frac{1}{|\Delta|L}\left\|\left(p_y - Bx - \frac{\Phi}{L}\right) P_\Delta(\Phi)\right\|_1 &\leqslant \frac{1}{|\Delta|L}\left\|\left(p_y - Bx - \frac{\Phi}{L}\right) P_\Delta(\Phi)\| \cdot \|P_\Delta(\Phi)\right\|_1 \\
&\leqslant \frac{\sqrt{2}}{|\Delta|L} \sup_{\|\psi\|=1} (\langle\psi|P_\Delta(\Phi)(H(\Phi)-V)P_\Delta(\Phi)|\psi\rangle)^{1/2} \mathrm{Tr}\, P_\Delta(\Phi) \\
&\leqslant \frac{\sqrt{3B}}{|\Delta|L} \mathrm{Tr}\, P_\Delta(\Phi) = O(1).
\end{aligned} \tag{1.10}$$

Here $\|\cdot\|_1$ and $\|\cdot\|$ are the trace and operator norms respectively and we used $\|AB\|_1 \leqslant \|A\| \cdot \|B\|_1$ for A bounded and B trace class. In the last equality we used there are $O(L)$ states in Δ because of (1.6) so that the final bound is uniform with respect to L. Since we have assumed that σ_e is independent of Φ, by averaging over Φ we get

$$\begin{aligned}
\sigma_e &= \lim_{L\to\infty} \frac{1}{|\Delta|}\int_0^{2\pi} \frac{\mathrm{d}\Phi}{2\pi} \sum_{E_k(\Phi)\in\Delta} \frac{\mathrm{d}E_k(\Phi)}{\mathrm{d}\Phi} \\
&= \lim_{L\to\infty} \frac{1}{|\Delta|} \sum_{k_{\min}}^{k_{\max}} \int_0^{2\pi} \frac{\mathrm{d}\Phi}{2\pi} \frac{\mathrm{d}E_k(\Phi)}{\mathrm{d}\Phi} \\
&= \lim_{L\to\infty} \frac{1}{2\pi|\Delta|} \sum_{k_{\min}}^{k_{\max}} (E_{k+1}(0) - E_k(0)) \\
&= \lim_{L\to\infty} \frac{1}{2\pi|\Delta|}\big(E_{k_{\max}} - E_{k_{\min}}\big) = \frac{1}{2\pi}.
\end{aligned} \tag{1.11}$$

For the first equality we use (1.3), (1.4) and dominated convergence. To obtain the second equality we consider separately the contributions of the eigenvalues with $k_{\min} \leqslant k \leqslant k_{\max}$ such that $E_k(\Phi) \in \Delta$ for all $\Phi \in [0, 2\pi]$, and of a finite number of eigenvalues with $k < k_{\min}$ (resp $k > k_{\max}$) which enter (resp leave) Δ as Φ varies from 0 to 2π. From (1.6) and (3.15) this latter contribution is $O(L^{-1})$. Finally (1.5) is used in the third equality. Here the units are such that $e = \hbar = 1$ so $\frac{1}{2\pi} = \frac{e^2}{h}$.

Section 2 contains the proofs of lemmas 1 and 2 and a third lemma that is needed for the proof of theorem 1 in section 3. The appendices A and B contain technical estimates.

2. Discreteness of edge spectrum and spectral flow

Proof of lemma 1. Let $D > 0$ to be chosen later (large) and $V_D(x, y) = V(x, y)$ for $x \leqslant -D$, $V_D(x, y) = 0$ for $x > -D$. Then $V(x, y) - V_D(x, y)$ has compact support and a standard argument using the resolvent identity implies that the essential spectra of

$$H_D(\Phi) = H_e(\Phi) + V_D(x, y) \tag{2.1}$$

and

$$H(\Phi) = H_D(\Phi) + V(x, y) - V_D(x, y) \tag{2.2}$$

coincide [11]. Therefore if we show that $\sigma(H_D(\Phi)) \cap \tilde{G}_0$ contains only isolated eigenvalues of finite multiplicity, the same is true for $H(\Phi)$. This will be achieved below using a decoupling scheme [12, 13] which proves that $\sigma(H_D(\Phi)) \cap \tilde{G}_0$ is a small perturbation of $\sigma(H_e(\Phi)) \cap \tilde{G}_0$. The set $\sigma(H_e(\Phi))$ consists of non-degenerate energy levels $\epsilon_n\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right)$, $n \in \mathbb{N}$ the Landau index and $m \in \mathbf{Z}$, where $\epsilon_n(k), k \in \mathbb{R}$ the wavenumber conjugate to y, are the spectral branches of $H_{e,\infty}$. These spectral branches are monotone increasing entire functions of k with $\epsilon_n(k) \to +\infty$ for $k \to +\infty$ and $\epsilon_n(k) \to \left(n + \frac{1}{2}\right)B$ for $k \to -\infty$ (see for example [3]).

In order to set up the decoupling scheme we introduce the characteristic functions $\chi_e(x)$ of $-\frac{D}{2} \leqslant x < +\infty$ and $\chi_b(x)$ of $-\infty \leqslant x < -\frac{D}{2}$. Note that $\chi_e(x) + \chi_b(x) = 1$ for all x. We also need the monotone and twice differentiable functions $J_e(x)$, $J_b(x)$ such that $J_e(x) = 0$ for $-\infty < x < -\frac{3D}{4} - 1$ and $J_e(x) = 1$ for $-\frac{3D}{4} + 1 < x < \infty$; $J_b(x) = 1$ for $-\infty < x < -\frac{D}{4} - 1$, $J_b(x) = 0$ for $-\frac{D}{4} + 1 < x < \infty$.

We introduce the Green functions $G_\alpha(z) = (H_\alpha(\Phi) - z)^{-1}$ for $\alpha = e, b, D$ and $z \in \mathbb{C}$ in the resolvent set of the corresponding Hamiltonian. Since

$$H_D(\Phi) J_\alpha = H_\alpha(\Phi) J_\alpha \qquad \text{for} \quad \alpha = e, b \tag{2.4}$$

following [13] we have

$$\begin{aligned}(H_D(\Phi) - z)(J_e G_e(z)\chi_e + J_b G_b(z)\chi_b) &= (H_e(\Phi) - z) J_e G_e(z)\chi_e + (H_b(\Phi) - z) J_b G_b(z)\chi_b \\ &= J_e\chi_e + J_b\chi_b + \tfrac{1}{2}\left[p_x^2, J_e\right] G_e(z)\chi_e + \tfrac{1}{2}\left[p_x^2, J_b\right] G_b(z)\chi_b \\ &= 1 + K_e(z) + K_b(z)\end{aligned} \tag{2.5}$$

where $K_\alpha(z) = \frac{1}{2}\left[p_x^2, J_\alpha\right] G_\alpha(z)\chi_\alpha$, $\alpha = e, b$. Thus

$$(H_D(\Phi) - z)^{-1} = (J_e G_e(z)\chi_e + J_b G_b(z)\chi_b)(1 + K_e(z) + K_b(z))^{-1}. \tag{2.6}$$

In appendix A, we prove the following estimates for the operator norms of $K_e(z)$ and $K_b(z)$ for $\frac{B}{2} + w < \operatorname{Re} z < \frac{3B}{2} - w$ (in what follows c is a generic positive numerical constant)

$$\|K_e(z)\| \leqslant \frac{cB^{\frac{3}{2}}L}{\delta_e(z)}\, \mathrm{e}^{-cBD^2} \tag{2.7}$$

$$\|K_b(z)\| \leqslant \frac{cB^{\frac{3}{2}}L}{\delta_0(z) - cw} \,\mathrm{e}^{-c\sqrt{B}D} \tag{2.8}$$

where $\delta_e(z) = \mathrm{dist}(z, \sigma(H_e(\Phi))$ and where $\delta_0(z) = \min\left(\left|z - \frac{B}{2}\right|, \left|z - \frac{3B}{2}\right|\right)$. We have to take w small enough so that the denominator in (2.8) stays positive. Later on we choose z appropriately and D large enough so that both terms become smaller than $\frac{1}{2}$. Thus

$$(H_D(\Phi) - z)^{-1} = J_e G_e(z)\chi_e + J_b G_b(z)\chi_b + R(z) \tag{2.9}$$

where

$$\|R(z)\| \leqslant (\|G_e(z)\| + \|G_b(z)\|)\,[(1 - \|K_e(z)\| - \|K_b(z)\|)^{-1} - 1]. \tag{2.10}$$

Let $m \in \mathbf{Z}$ be such that $\epsilon_0\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right)$ is an eigenvalue belonging to $\sigma(H_e(\Phi)) \cap \tilde{G}_0$. We can choose $\rho > 0$ small enough independent of m and L such that the circle C_m with centre $\epsilon_0\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right)$ and radius $\frac{\rho}{L}$ encloses only one such eigenvalue. By choosing z in a sufficiently thin annulus around C_m and D large enough, (2.7) and (2.8) can be made smaller than $\frac{cB^{\frac{3}{2}}L^2}{\rho}\,\mathrm{e}^{-c\sqrt{B}D} < \frac{1}{2}$. At the same time from (2.10) we have

$$\|R(z)\| \leqslant \frac{cB^{\frac{3}{2}}L^3}{\rho^2}\,\mathrm{e}^{-c\sqrt{B}D} \tag{2.11}$$

so that from (2.9) $(H_D(\Phi) - z)^{-1}$ is well defined for z in a thin annulus surrounding C_m. Therefore, we can compute the spectral projection $P_D(m, \Phi)$ of $H_D(\Phi)$ for the interval $I_m = \left]\epsilon_0\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right) - \frac{\rho}{L}, \epsilon_0\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right) + \frac{\rho}{L}\right[$ by Cauchy's formula. Let $P_e(m, \Phi)$ be the projector of $H_e(\Phi)$ corresponding to the level $\epsilon_0\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right)$. Thanks to (2.9), (2.11) we obtain for D large enough

$$\|P_D(m, \Phi) - P_e(m, \Phi)\| \leqslant \frac{cB^{\frac{3}{2}}L^2}{\rho}\,\mathrm{e}^{-c\sqrt{B}D} < 1. \tag{2.12}$$

This estimate implies that $\sigma(H_D(\Phi)) \cap I_m$ contains only one eigenvalue of multiplicity equal to 1. Note that this conclusion holds for all $I_m \subset \tilde{G}_0$. Finally, since $H_e(\Phi)$ and $H_b(\Phi)$ have no spectrum in $(\cup_m I_m)^c \cap \tilde{G}_0$ we deduce from (2.7), (2.8), (2.9) that $H_D(\Phi, L)$ has no spectrum in that same set. Therefore, $\sigma(H_D(\Phi, L)) \cap \tilde{G}_0$ consists of isolated eigenvalues of multiplicity 1.

It remains to be shown that an eigenvalue $E_k(0) \in \tilde{G}_0$ can be continued into one or several analytic branches $E_k(\Phi)$ for Φ small enough. In the present case it is sufficient to show [11] that $(py - Bx)$ is relatively bounded with respect to $H(0)$. For any ψ in the domain of $H(0)$ and any complex number z with $\mathrm{Im}\, z \neq 0$ we have

$$\begin{aligned}
\tfrac{1}{2}\|(p_y - Bx)\psi\|^2 &\leqslant \langle\psi|(H(0) - V)\psi\rangle \\
&= \langle\psi|(H(0) - z)^{-1}(H(0) - z)|(H(0) - \bar{z} + z)\psi\rangle - \langle\psi|V\psi\rangle \\
&\leqslant \|(H(0) - z)^{-1}\| \cdot \|H(0)\psi\|^2 + |z| \cdot \|\psi\|^2 \\
&\quad + |z|^2\|(H(0) - z)^{-1}\| \cdot \|\psi\|^2 + w\|\psi\|^2 \\
&\leqslant \frac{1}{|\mathrm{Im}\, z|}\|H(0)\psi\|^2 + \left(|z| + \frac{|z|^2}{|\mathrm{Im}\, z|} + w\right)\|\psi\|^2.
\end{aligned} \tag{2.13}$$

This concludes the proof of the lemma. □

Remark. In (2.13) we can take $|\mathrm{Im}\, z|$ as large as we wish so the size of the interval of analyticity is not limited by the relative bound but rather by the fact that the branch $E_k(\Phi)$ may merge in the Landau bands (outside of G_0) where it may not be isolated anymore. Inequality (3.15) shows that for L large enough the maximal variation of $E_k(\Phi)$ is $2\pi\sqrt{3B}L^{-1}$, so that if $E_k(\Phi)$ is contained in Δ for some Φ then it is contained in $\tilde{G}_0$, and it is analytic for all $\Phi\in[0,2\pi]$.

Before presenting the formal proof of lemma 2 we would like to point out that in fact (1.2) is closely related to the ideas in [3] and [4]. Using the unitary translation operator $x\to x+\frac{\Phi}{BL}$ and the Feynman–Hellman theorem it is easy to see that

$$L\frac{\mathrm{d}}{\mathrm{d}\Phi}E_k(\Phi)=\langle\Psi_k(\Phi)|(W'+\partial_x V)\Psi_k(\Phi)\rangle$$

where $|\Psi_k(\Phi)\rangle$ is the eigenstate with eigenvalue $E_k(\Phi)$. Using the methods of [3] or [4] one may show that for $E_k(\Phi)\in\Delta$, $|\Psi_k(\Phi)\rangle$ is mainly concentrated near the region where $W'(x)$ is large so that (1.2) holds provided both V, $\partial_x V$ are small enough. Here we follow a different method which is closer to the original argument of Halperin [2] in that it uses directly the relation (1.4) instead of (2.13). Only the smallness of V is required.

Proof of lemma 2. The eigenstates $|u_{nm}(\Phi)\rangle$ of $H_e(\Phi)$ with eigenvalues $\epsilon_n\left(\frac{2\pi m}{L}+\frac{\Phi}{L}\right)$ are of the form

$$\langle xy|u_{nm}(\Phi)\rangle=\mathrm{e}^{\mathrm{i}\frac{2\pi m}{L}y}h_{nm}(x) \tag{2.14}$$

so that $\left\langle u_{nm}(\Phi)\middle|\left(p_y-Bx-\frac{\Phi}{L}\right)u_{n'm'}(\Phi)\right\rangle=0$ for $m\neq m'$ and all n,n'. Therefore writing

$$|\Psi_k(\Phi)\rangle=\left|\Psi_k^0(\Phi)\right\rangle+\left|\Psi_k^1(\Phi)\right\rangle \tag{2.15}$$

where

$$\left|\Psi_k^0(\Phi)\right\rangle=\sum_{m=-\infty}^{+\infty}c_k^{0m}|u_{0m}(\Phi)\rangle \tag{2.16}$$

$$\left|\Psi_k^1(\Phi)\right\rangle=\sum_{n\geqslant 1}\sum_{m=-\infty}^{+\infty}c_k^{nm}|u_{nm}(\Phi)\rangle \tag{2.17}$$

we obtain from (1.3), (1.4)

$$\begin{aligned}L\frac{\mathrm{d}}{\mathrm{d}\Phi}E_k(\Phi)=&\sum_{m=-\infty}^{+\infty}\left|c_k^{0m}\right|^2\left\langle u_{0m}(\Phi)\middle|\left(p_y-Bx-\frac{\Phi}{L}\right)u_{0m}(\Phi)\right\rangle\\&+2\mathrm{Re}\left\langle\Psi_k^0(\Phi)\middle|\left(p_y-Bx-\frac{\Phi}{L}\right)\Psi_k^1(\Phi)\right\rangle\\&+\left\langle\Psi_k^1(\Phi)\middle|\left(p_y-Bx-\frac{\Phi}{L}\right)\Psi_k^1(\Phi)\right\rangle.\end{aligned} \tag{2.18}$$

First we show that the last two terms on the right-hand side of (2.18) are bounded by the norm $\sqrt{3B}\left\|\Psi_k^1(\Phi)\right\|$. The Schwartz inequality implies

$$\begin{aligned}\left|\left\langle\Psi_k^0(\Phi)\middle|\left(p_y-Bx-\frac{\Phi}{L}\right)\Psi_k^1(\Phi)\right\rangle\right|&\leqslant\left\|\Psi_k^1(\Phi)\|\cdot\|\left(p_y-Bx+\frac{\Phi}{L}\right)\Psi_k^0(\Phi)\right\|\\&\leqslant\sqrt{2}\left\|\Psi_k^1(\Phi)\right\|\left(\left\langle\Psi_k^0(\Phi)\middle|H_e(\Phi)\Psi_k^0(\Phi)\right\rangle\right)^{1/2}\\&\leqslant\sqrt{2}\left\|\Psi_k^1(\Phi)\right\|\left(\left\langle\Psi_k^0(\Phi)\middle|H_e(\Phi)\Psi_k^0(\Phi)\right\rangle+\left\langle\Psi_k^1(\Phi)\middle|H_e(\Phi)\Psi_k^1(\Phi)\right\rangle\right)^{1/2}\\&=\sqrt{2}\left\|\Psi_k^1(\Phi)\right\|\left(\left\langle\Psi_k(\Phi)\middle|H_e(\Phi)\Psi_k(\Phi)\right\rangle\right)^{1/2}\\&\leqslant\sqrt{2}\left\|\Psi_k^1(\Phi)\right\|(E_k(\Phi)+w)^{1/2}\leqslant\sqrt{3B}\left\|\Psi_k^1(\Phi)\right\|.\end{aligned} \tag{2.19}$$

For the third matrix element on the right-hand side of (2.18) the same method leads to an identical estimate. From the Feynman–Hellman formula we have

$$\left\langle u_{0m}(\Phi) \left| \left(p_y - Bx - \frac{\Phi}{L} \right) u_{0m}(\Phi) \right\rangle \right. = L \frac{\mathrm{d}}{\mathrm{d}\Phi} \epsilon_0 \left(\frac{2\pi m}{L} + \frac{\Phi}{L} \right)$$
$$= \epsilon_0' \left(\frac{2\pi m}{L} + \frac{\Phi}{L} \right) \tag{2.20}$$

where $\epsilon_0'(k)$ is the derivative of the lowest monotone increasing spectral branch corresponding to the Hamiltonian $H_{e,\infty}$. From (2.18), (2.19), (2.20)

$$L \frac{\mathrm{d}}{\mathrm{d}\Phi} E_k(\Phi) \geqslant \sum_{m=-\infty}^{+\infty} \left| c_k^{0m} \right|^2 \epsilon_0' \left(\frac{2\pi m}{L} + \frac{\Phi}{L} \right) - 2\sqrt{3B} \left\| \Psi_k^1(\Phi) \right\|$$
$$\geqslant v_F(M) \sum_{|m-M| \leqslant \bar{m}} \left| c_k^{0m} \right|^2 - 2\sqrt{3B} \left\| \Psi_k^1(\Phi) \right\| \tag{2.21}$$

with the Fermi velocity

$$v_F(M) = \min_{|m-M| \leqslant \bar{m}} \epsilon_0' \left(\frac{2\pi m}{L} + \frac{\Phi}{L} \right). \tag{2.22}$$

The integers M and $\bar{m}$ will be chosen conveniently below. Writing the Schrödinger equation in the form

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{+\infty} c_k^{nm} \left(\epsilon_n \left(\frac{2\pi m}{L} + \frac{\Phi}{L} \right) - E_k(\Phi) \right) |u_{nm}(\Phi)\rangle = V(x, y) |\Psi_k(\Phi)\rangle \tag{2.23}$$

and taking the norm on both sides

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{+\infty} \left| c_k^{nm} \right|^2 \left(\epsilon_n \left(\frac{2\pi m}{L} + \frac{\Phi}{L} \right) - E_k(\Phi) \right)^2 \leqslant w^2. \tag{2.24}$$

Dropping the term $n = 0$, using $\left(\epsilon_n\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right) - E_k(\Phi)\right)^2 \geqslant \left(\frac{B}{2} - \delta\right)^2$ for $n \geqslant 1$ and $E_k(\Phi) \in \Delta$ we get

$$\sum_{n \geqslant 1} \sum_{m=-\infty}^{\infty} \left| c_k^{nm} \right|^2 = \left\| \Psi_k^1(\Phi) \right\|^2 \leqslant \frac{w^2}{\left(\frac{B}{2} - \delta\right)^2}. \tag{2.25}$$

From (2.24) one can also derive a lower bound for $\sum_{|m-M|<\bar{m}} \left| c_k^{0m} \right|^2$. Indeed retaining only the term $n = 0$ and using the monotonicity of $\epsilon_0\left(\frac{2\pi m}{L} + \frac{\Phi}{L}\right)$ we have

$$A(M, \bar{m})^2 \sum_{|m-M| > \bar{m}} \left| c_k^{0m} \right|^2 \leqslant w^2 \tag{2.26}$$

where $A(M, \bar{m})$ is the smallest of the two numbers $\left|\epsilon_0\left(\frac{2\pi}{L}(M \pm \bar{m}) + \frac{\Phi}{L}\right)\right| - E_k(\Phi)\Big|$. Now we choose any M such that $\epsilon_0\left(\frac{2\pi M}{L} + \frac{\Phi}{L}\right) \in \Delta$ and since $E_k(\Phi) \in \Delta$ we can take $\bar{m}$ such that $A(M, \bar{m}) \geqslant \frac{B}{2} - 2\delta$. Thus

$$\sum_{|m-M| > \bar{m}} \left| c_k^{0m} \right|^2 \leqslant \frac{w^2}{\left(\frac{B}{2} - 2\delta\right)^2}. \tag{2.27}$$

Finally the normalization condition for $|\Psi_k(\Phi)\rangle$ combined with (2.25) and (2.27) implies

$$\sum_{|m-M| \leqslant \bar{m}} \left| c_k^{0m} \right|^2 \geqslant 1 - \frac{2w^2}{\left(\frac{B}{2} - 2\delta\right)^2} \tag{2.28}$$

From (2.21), (2.25) and (2.28) we have

$$L\frac{\mathrm{d}}{\mathrm{d}\Phi}E_k(\Phi) \geqslant v_F(M)\left[1 - 2\left(1 + \frac{\sqrt{3B}}{v_F(M)}\right)\frac{w^2}{\left(\frac{B}{2} - 2\delta\right)^2}\right]. \tag{2.29}$$

Clearly $v_F(M)$ is a strictly positive number which does not depend on V but only on W and B. Therefore (2.29) implies the result of the lemma for w and δ small enough. □

It will become clear in the next section that the proof of theorem 1 requires the absence of crossings for the branches $E_k(\Phi)$ in Δ. Since we do not know *a priori* if this is true for $H(\Phi)$, an intermediate step is to construct a suitable perturbation of $H(\Phi)$ for which the non-crossing property is satisfied. The perturbation that is added here has the effect of lifting the degeneracy at each crossing in Δ in a way that (1.2) still holds for the perturbed branches. This is the content of the next lemma.

Lemma 3. *Fix B, w, δ and L as in lemma 2. Assume that $V(x, y)$ is such that the eigenvalues $E_l(0)$ are not degenerate. One can construct a finite rank perturbation $R(\Phi)$ with $\|R(\Phi)\| \leqslant L^{-10}$ such that the spectrum of $\tilde{H}(\Phi) = H(\Phi) + R(\Phi)$ in Δ consists of non-degenerate eigenvalues forming infinitely differentiable spectral branches which do not cross and are labelled as $\tilde{E}_l(\Phi)$ with $\tilde{E}_l(0) = E_l(0)$. Moreover the new branches satisfy*

$$L\frac{\mathrm{d}}{\mathrm{d}\Phi}\tilde{E}_l(\Phi) \geqslant \tilde{\alpha} \tag{2.30}$$

where $\tilde{\alpha}$ is strictly positive and independent of L.

Proof of lemma 3. Let $P_\Delta(\Phi)$ be the eigenprojector of $H(\Phi)$ onto Δ. Then we have

$$P_\Delta(\Phi)H(\Phi)P_\Delta(\Phi) = \sum_{E_l(\Phi)\in\Delta} E_l(\Phi)|\Psi_l(\Phi)\rangle\langle\Psi_l(\Phi)|. \tag{2.31}$$

Since the branches $E_l(\Phi)$ are analytic and the eigenvalues are not degenerate for $\Phi = 0$, the possible crossings are necessarily isolated. Indeed if two branches coincided on a set with accumulation points they would concide over the whole interval $[0, 2\pi]$ and therefore violate the non-degeneracy assumption at $\Phi = 0$. Therefore, we can assume without loss of generality that there is at most a finite number of crossings in Δ. Let us construct the perturbation $R(\Phi)$. First consider the set $\mathcal{C}$ of pairs of branches which cross in Δ (note that n branches may cross at the same point and contribute as $\frac{n(n-1)}{2}$ pairs). Pick one pair of branches in $\mathcal{C}$, say (ij), and assume $E_i(0) < E_j(0)$. Suppose they cross at points Φ_{ij}^μ where the label μ takes into account the fact that the branches i and j may cross more than once, i.e.

$$E_i\left(\Phi_{ij}^\mu\right) = E_j\left(\Phi_{ij}^\mu\right). \tag{2.32}$$

Let $\lambda_{ij}^\mu(\Phi)$ be infinitely differentiable test functions centred at Φ_{ij}^μ with a compact support of width β_1 and $\max_{0\leqslant\Phi\leqslant 2\pi}\left|\lambda_{ij}^\mu(\Phi)\right| \leqslant \lambda_1$. The real numbers δ_1 and λ_1 will be adjusted in a suitable way below. Add to the Hamiltonian $H(\Phi)$ the perturbation

$$R_1(\Phi) = \sum_\mu \lambda_{ij}^\mu(\Phi)(|\Psi_i(\Phi)\rangle\langle\Psi_j(\Phi)| + |\Psi_j(\Phi)\rangle\langle\Psi_i(\Phi)|). \tag{2.33}$$

We take β_1 small enough so that supports of the test functions do not contain $\Phi = 0$ and do not overlap. In order to diagonalize the new Hamiltonian it is sufficient to work in the two-dimensional subspace of the branches i and j. The spectral branches of the new Hamiltonian do not change for $k \neq i, j$, whereas for $k = i, j$ they become

$$E_i^1(\Phi) = \tfrac{1}{2}\left(E_i(\Phi) + E_j(\Phi) - \sqrt{(E_i(\Phi) - E_j(\Phi))^2 + \lambda_{ij}^\mu(\Phi)^2}\right) \tag{2.34}$$

and

$$E_j^1(\Phi) = \frac{1}{2}\left(E_i(\Phi) + E_j(\Phi) + \sqrt{(E_i(\Phi) - E_j(\Phi))^2 + \lambda_{ij}^{\mu}(\Phi)^2}\right). \tag{2.35}$$

Since the difference

$$E_j^1(\Phi) - E_i^1(\Phi) = \sqrt{(E_i(\Phi) - E_j(\Phi))^2 + \lambda_{ij}^{\mu}(\Phi)^2} \tag{2.36}$$

is always strictly positive, the new pair (ij) is non-degenerate for all values of Φ. Moreover by choosing λ_1 small enough we can make sure that we do not introduce more crossings. Therefore, the perturbed Hamiltonian

$$H_1(\Phi) = H(\Phi) + R_1(\Phi) \tag{2.37}$$

has a new set $\mathcal{C}_1$ of pairs of branches which cross with one element less than $\mathcal{C}$. One can construct in the same way a perturbation $R_2(\Phi)$ of (2.37) (with δ_2, λ_2 small enough) so that the new Hamiltonian $H_2(\Phi) = H_1(\Phi) + R_2(\Phi)$ has two less pairs of branches which cross than $H(\Phi)$. Since there is at most a finite number of such pairs by iterating this construction we end up with the Hamiltonian

$$\tilde{H}(\Phi) = H(\Phi) + \sum_p R_p(\Phi) = H(\Phi) + R(\Phi) \tag{2.38}$$

of the lemma, where the sum over p contains a finite number of terms. Note that $\tilde{H}(0) = H(0)$ so that the labelling of the lemma holds. The norm of the total perturbation is

$$\|R(\Phi)\| \leqslant \sum_p \|R_p(\Phi)\| \leqslant \sum_p \lambda_p. \tag{2.39}$$

The condition $\|R(\Phi)\| \leqslant L^{-10}$ can always be achieved by choosing at each step

$$\lambda_p \leqslant \frac{\beta_p}{L^{10+p}} \tag{2.40}$$

and $\beta_p \leqslant \frac{1}{10}$.

It remains to check that (2.30) holds. From the formulae (2.34), (2.35) and lemma 2, it is easy to check that at the first step of the construction the new branches have new derivatives satisfying

$$\frac{\mathrm{d}}{\mathrm{d}\Phi} E_{i,j}^1(\Phi) \geqslant \min\left(\frac{\mathrm{d}}{\mathrm{d}\Phi} E_i(\Phi), \frac{\mathrm{d}}{\mathrm{d}\Phi} E_j(\Phi)\right) - \frac{1}{2}\left|\frac{\mathrm{d}}{\mathrm{d}\Phi}\lambda_{ij}^{\mu}(\Phi)\right| \tag{2.41}$$

for all Φ. At each step of the construction it is possible to choose test functions such that

$$\max_{0\leqslant\Phi\leqslant 2\pi}\left|\frac{\mathrm{d}}{\mathrm{d}\Phi}\lambda_{ij}^{\mu}(\Phi)\right| \leqslant \frac{2}{L^{10+p}} \tag{2.42}$$

in a way consistent with (2.40). So at the first step $(p = 1)$

$$\frac{\mathrm{d}}{\mathrm{d}\Phi} E_{i,j}^1(\Phi) \geqslant \frac{\alpha}{L} - \frac{1}{L^{11}}. \tag{2.43}$$

Of course (2.43) is also valid for the spectral branches of $H_1(\Phi)$ that correspond to $k \in \mathcal{N}$. Therefore it is valid for all eigenvalues of $H_1(\Phi)$. By iterating the construction we see that any branch of (2.38) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\Phi}\tilde{E}_l(\Phi) \geqslant \frac{\alpha}{L} - \sum_p \frac{1}{L^{10+p}} \tag{2.44}$$

which implies (2.30). □

3. Relative index and level spacing

The main goal of this section is to prove theorem 1. Let us first outline the strategy of the proof. Without loss of generality we can suppose that V is such that $E_k(0)$ are non-degenerate. Indeed if this is not the case one may find a sufficiently small perturbation $u(x, y)$, $\|u\|_\infty < L^{-10}$ such that this hypothesis is satisfied for $V + u$. If (1.5), (1.6) hold for $V + u$ then they hold for V because the perturbation of the discrete levels separated by $O(L)$ is at most $O(L^{-10})$. From lemma 2, we know that for $E_k(\Phi) \in \Delta$ there is a non-trivial spectral flow: the branches are monotone increasing, and since $H(0)$ and $H(2\pi)$ are unitarily equivalent we must have $E_k(2\pi) = E_{k'}(0)$, $k' > k$. We want to show that in fact $k' = k + 1$. Let E_F be a single 'Fermi energy' lying between two consecutive levels of both Hamiltonians $H_D(0)$ and $\tilde{H}(0)$. Define the integers Q_F^D and $\tilde{Q}_F$ to be the number of branches of the corresponding Hamiltonians which cross E_F as Φ varies from 0 to 2π. We will show that $Q_F^D = \tilde{Q}_F = 1$. We know from lemma 3 that the branches of $\tilde{H}(\Phi)$ do not have crossings and from the proof of lemma 1 that the same is true for the branches of $H_D(\Phi)$. This enables us to relate $\tilde{Q}_F$ and Q_F^D to the notion of relative index of a pair of projections introduced by Avron *et al* [14]. Then by using the fact that the Fredholm index of an operator does not change under compact perturbations we deduce that $\tilde{Q}_F = Q_F^D$. By explicit computation we can check that $Q_F^D = 1$ and therefore $\tilde{Q}_F = 1$ which implies that $\tilde{E}_k(2\pi) = \tilde{E}_{k+1}(0)$. Since the branches of $\tilde{H}(\Phi)$ are a small perturbation of those of $H(\Phi)$ we deduce (1.5). Estimate (1.6) is then an immediate consequence.

In order to make the paper self-contained we give a short summary of the mathematical tools used below, as developed in [14]. Let P and Q be orthogonal projections on a separable Hilbert space $\mathcal{H}$. The pair $(P; Q)$ is called Fredholm if QP viewed as a map from $P\mathcal{H}$ to $Q\mathcal{H}$ is a Fredholm operator. The relative index $\mathrm{Ind}(P; Q)$ of the pair is the usual Fredholm index of $T = QP$, that is $\dim \mathrm{Ker}(T^\dagger T) - \dim \mathrm{Ker}(TT^\dagger)$. One proves that $(P; Q)$ is a Fredholm pair if and only if 1 and -1 are isolated finitely degenerate eigenvalues of $P - Q$, when they belong to the spectrum. Moreover one has $\mathrm{Ind}(P, Q) = \dim \mathrm{Ker}(P - Q - 1) - \dim \mathrm{Ker}(P - Q + 1)$. A useful formula (we use it for $m = 0$) states that if $(P - Q)^{2m+1}$ is trace class for some integer m then $(P; Q)$ is a Fredholm pair and $\mathrm{Ind}(P; Q) = \mathrm{Tr}(P - Q)^{2n+1}$, for all $n \geqslant m$. A central result on which we rely is that if $(P; Q)$ and $(Q; R)$ are Fredholm pairs and either $P - Q$ or $Q - R$ is compact then $(P; R)$ is a Fredholm pair and

$$\mathrm{Ind}(P; R) = \mathrm{Ind}(P; Q) + \mathrm{Ind}(Q; R). \tag{3.1}$$

Finally we note that if $(P; Q)$ is Fredholm then so is $(UPU^\dagger; UQU^\dagger)$ for any unitary U and the relative index remains invariant. Also $\mathrm{Ind}(P; Q) = -\mathrm{Ind}(Q; P)$.

3.1. Relation between $\tilde{Q}_F$, Q_F^D and the relative index of a pair of projections

We fix $E_F \in \Delta$ between two consecutive levels of $\tilde{H}(0)$ and $\tilde{H}(2\pi)$ (recall that they have the same spectrum). Let $\tilde{P}_{F,0}$ (resp. $\tilde{P}_{F,2\pi}$) be the projectors of $\tilde{H}(0)$ (resp. $\tilde{H}(2\pi)$) onto the energy range $]-\infty, E_F]$. We also need the projector on levels $\tilde{E}_k(0)$ whose spectral branch $\tilde{E}_k(\Phi)$ crosses E_F. Namely

$$\tilde{P}^c_{F,0} = \sum_{\tilde{E}_k(0) < E_F \,\mathrm{s.t}\, \tilde{E}_k(\Phi) \,\mathrm{crosses}\, E_F} P(\tilde{E}_k(0)) \tag{3.2}$$

where $P(\tilde{E}_k(0))$ is the eigenprojector of $\tilde{H}(0)$ corresponding to the discrete level $\tilde{E}_k(0)$. Since $E_F \in \Delta$ by taking L large enough we are assured that this sum is finite and that the branches crossing E_F remain in Δ for all $\Phi \in [0, 2\pi]$.

Setting $\tilde{P}^{n.c}_{F,0} = \tilde{P}_{F,0} - \tilde{P}^{c}_{F,0}$ we have

$$\tilde{Q}_F = \mathrm{Tr}\, \tilde{P}^{c}_{F,0} = \mathrm{Tr}\left(\tilde{P}_{F,0} - \tilde{P}^{nc}_{F,0}\right) = \mathrm{Ind}\left(\tilde{P}_{F,0};\ \tilde{P}^{nc}_{F,0}\right). \tag{3.3}$$

We introduce a smooth, monotone increasing function of time $\varphi(t)$, $0 \leqslant t \leqslant T$, $\varphi(0) = 0$ and $\varphi(T) = 2\pi$, describing the adiabatic switching of a flux quantum through the axis of the cylinder. Let U_t be the unitary time evolution associated with the time-dependent Hamiltonian $\tilde{H}(\varphi(t))$. From lemma 3, as t varies the spectral branches in Δ do not cross and are monotone increasing. So an application of the adiabatic theorem [15] assures that $U_T \tilde{P}^{nc}_{F,0} U^{\dagger}_T$ tends to $\tilde{P}_{F,2\pi}$. Thus there exists some large enough T_0 such that for $T > T_0$, the pair of projections $\left(\tilde{P}^{nc}_{F,0};\ U^{\dagger}_T \tilde{P}_{F,2\pi} U_T\right)$ satisfies

$$\left\| \tilde{P}^{nc}_{F,0} - U^{\dagger}_T \tilde{P}_{F,2\pi} U_T \right\| < 1. \tag{3.4}$$

Thus it is Fredholm and $\mathrm{Ind}\left(\tilde{P}^{nc}_{F,0};\ U^{\dagger}_T \tilde{P}_{F,2\pi} U_T\right) = 0$. Since $\tilde{P}_{F,0} - \tilde{P}^{nc}_{F,0}$ is finite rank we can apply (3.1) to get

$$\begin{aligned}\tilde{Q}_F &= \mathrm{Ind}\left(\tilde{P}_{F,0};\ \tilde{P}^{nc}_{F,0}\right)\\ &= \mathrm{Ind}\left(\tilde{P}_{F,0};\ U^{\dagger}_T \tilde{P}_{F,2\pi} U_T\right) + \mathrm{Ind}\left(U^{\dagger}_T \tilde{P}_{F,2\pi} U_T;\ \tilde{P}^{nc}_{F,0}\right)\\ &= \mathrm{Ind}\left(\tilde{P}_{F,0};\ U^{\dagger}_T \tilde{P}_{F,2\pi} U_T\right).\end{aligned} \tag{3.5}$$

Finally let U be the multiplication operator by $\mathrm{e}^{\mathrm{i}\frac{2\pi}{L}y}$. Since U does not change the boundary conditions and $U^{\dagger} H(0) U = H(2\pi)$ we obtain the formula

$$\tilde{Q}_F = \mathrm{Ind}\left(\tilde{P}_{F,0};\ U^{\dagger}_T U^{\dagger} \tilde{P}_{F,0} U U_T\right). \tag{3.6}$$

The same construction for $H_D(0)$ leads to

$$Q^D_F = \mathrm{Ind}\left(P^D_{F,0};\ U^{D\dagger}_T U^{\dagger} P^D_{F,0} U U^D_T\right) \tag{3.7}$$

where $P^D_{F,0}$ is the projector of $H_D(0)$ onto $]-\infty, E_F]$, and U^D_t is the time evolution associated with the Hamiltonian $H_D(\varphi(t))$. We remark that the identities of this paragraph can be checked by explicit computation for the simple toy Hamiltonian (1.7).

Remark. In [18] a different relative index for an infinite two-dimensional system is studied and related to the Hall conductivity viewed as a Chern number. It would be interesting to investigate the analogous relationship in the present case with a boundary.

3.2. Equality of $\tilde{Q}_F$ and Q^D_F

Since $V - V_D$ has a finite support, $(z - H_D(0))^{-1}(V - V_D)$ is a compact operator for z not in $\sigma(H_D(0))$. Therefore the resolvent identity and Cauchy's formula imply that $\tilde{P}_{F,0} - P^D_{F,0}$ is compact. Thus the pair $\left(\tilde{P}_{F,0};\ P^D_{F,0}\right)$ is Fredholm and we can apply (3.1) to get

$$\begin{aligned}\mathrm{Ind}\left(\tilde{P}_{F,0};\ U^{\dagger}_T U^{\dagger} \tilde{P}_{F,0} U U_T\right) &= \mathrm{Ind}\left(\tilde{P}_{F,0};\ P^D_{F,0}\right) + \mathrm{Ind}\left(P^D_{F,0};\ U^{\dagger}_T U^{\dagger} \tilde{P}_{F,0} U U_T\right)\\ &= \mathrm{Ind}\left(\tilde{P}_{F,0};\ P^D_{F,0}\right) + \mathrm{Ind}\left(P^D_{F,0};\ U^{\dagger}_T U^{\dagger} P^D_{F,0} U U_T\right)\\ &\quad + \mathrm{Ind}\left(U^{\dagger}_T U^{\dagger} P^D_{F,0} U U_T;\ U^{\dagger}_T U \tilde{P}_{F,0} U^{\dagger} U_T\right).\end{aligned} \tag{3.8}$$

The first and third terms in the last equality of (3.8) cancel. Thus

$$\begin{aligned}\tilde{Q}_F &= \mathrm{Ind}\left(P^D_{F,0};\ U^{\dagger}_T U^{\dagger} P^D_{F,0} U U_T\right)\\ &= \mathrm{Ind}\left(P^D_{F,0} U U_T P^D_{F,0} \middle| P^D_{F,0}\mathcal{H} \to P^D_{F,0}\mathcal{H}\right)\end{aligned} \tag{3.9}$$

where in the last line we introduced the Fredholm index of $P_{F,0}^D U U_T P_{F,0}^D$ viewed as a map from $P_{F,0}^D \mathcal{H}$ to itself ($\mathcal{H}$ the Hilbert space of the cylinder). From Dyson's equation

$$P_{F,0}^D U U_T P_{F,0}^D - P_{F,0}^D U U_T^D P_{F,0}^D = \int_0^T \mathrm{d}s\, P_{F,0}^D U U_{T-s}^D (V - V_D) U_s P_{F,0}^D. \tag{3.10}$$

Therefore, the Hilbert–Schmidt norm of the left-hand side is smaller than

$$\int_0^T \mathrm{d}s \left\| P_{F,0}^D U U_{T-s}^D (V - V_D) \right\|_{HS} \tag{3.11}$$

which is shown to be finite in appendix B. Thus the difference (3.10) is compact, and the two operators have the same Fredholm index

$$\mathrm{Ind}\left(P_{F,0}^D U U_T P_{F,0}^D \,\middle|\, P_{F,0}^D \mathcal{H} \to P_{F,0}^D \mathcal{H}\right) = \mathrm{Ind}\left(P_{F,0}^D U U_T^D P_{F,0}^D \,\middle|\, P_{F,0}^D \mathcal{H} \to P_{F,0}^D \mathcal{H}\right) \tag{3.12}$$

which is equivalent to $\tilde{Q}_F = Q_F^D$.

3.3. End of proof of (1.5) and (1.6)

From the analysis of section 2, we know that for D large enough (say $D = O(L)$) the branches of $H_e(\Phi)$ and $H_D(\Phi)$ that belong to Δ lie close to each other within a distance $O(\mathrm{e}^{-c\sqrt{BL}})$. Since the spacing of the branches of $H_e(\Phi)$ is $O(L^{-1})$ it follows that $Q_F^D = 1$ and therefore $\tilde{Q}_F = 1$. Thus $\tilde{E}_k(2\pi) = \tilde{E}_{k+1}(0)$ and since there exists $0 \leqslant \bar{\Phi} \leqslant 2\pi$ such that

$$\tilde{E}_k(2\pi) - \tilde{E}_k(0) = 2\pi \frac{\mathrm{d}\tilde{E}_k}{\mathrm{d}\Phi}(\bar{\Phi}) \tag{3.13}$$

from (2.30) we get the lower bound

$$|\tilde{E}_{k+1}(0) - \tilde{E}_k(0)| \geqslant \frac{2\pi\tilde{\alpha}}{L}. \tag{3.14}$$

Because $\tilde{E}_l(0) = E_l(0)$, this bound shows that the levels of $H(0)$ (or $H(2\pi)$) are spaced by $O(L^{-1})$. Using the spectral flow of $\tilde{H}(\Phi)$, together with the facts that the levels of $\tilde{H}(\Phi)$ and $H(\Phi)$ are separated by $O(L^{-10})$ and that $\frac{\mathrm{d}E_k(\Phi)}{\mathrm{d}\Phi}$ is strictly positive, one deduces that necessarily $E_k(2\pi) = E_{k+1}(0)$. Then proceeding as in (3.13) and (3.14) we obtain the lower bound (1.6). Finally the upper bound is a consequence of

$$\begin{aligned} L\left|\frac{\mathrm{d}E_k}{\mathrm{d}\Phi}(\bar{\Phi})\right| &= \langle \Psi_k(\bar{\Phi}) | p_y - Bx + \frac{\bar{\Phi}}{L} | \Psi_k(\bar{\Phi}) \rangle \\ &\leqslant \left\| \Psi_k(\bar{\Phi}) \| \cdot \| \left(p_y - Bx + \frac{\bar{\Phi}}{L} \right) \Psi_k(\bar{\Phi}) \right\| \\ &\leqslant (\langle \Psi_k(\bar{\Phi}) | 2H(\bar{\Phi}) | \Psi_k(\bar{\Phi}) \rangle - \langle \Psi_k(\bar{\Phi}) | 2V | \Psi_k(\bar{\Phi}) \rangle)^{\frac{1}{2}} \\ &\leqslant (2E_k(\bar{\Phi}) + 2w)^{\frac{1}{2}} \leqslant (3B)^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

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Appendix A

We start with a sketch of preliminary estimates for the Green function of the pure magnetic problem on the cylinder of circumference L,

$$H_0(\Phi) = \frac{1}{2} p_x^2 + \frac{1}{2} \left(p_y - Bx + \frac{\Phi}{L} \right)^2. \tag{A.1}$$

Using the spectral decomposition of the Green function $G_0(z) = (H_0(\Phi) - z)^{-1}$ on a basis of eigenfunctions

$$\mathrm{e}^{\mathrm{i}\frac{2\pi m}{L}y}\varphi_{n,m}(x) \tag{A.2}$$

and the Poisson summation formula we obtain

$$\langle x, y|G_0(\Phi)|x', y'\rangle = \sum_{m=-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\frac{\Phi}{L}(y-y'-mL)}\langle x, y - mL|G_{0,\infty}(z)|x', y'\rangle \tag{A.3}$$

where $G_{0,\infty}(z)$ is the Green function of the pure magnetic problem on the infinite two-dimensional plane. In the Landau gauge ($\mathbf{r} = (x, y)$)

$$\langle \mathbf{r}|G_{0,\infty}(z)|\mathbf{r}'\rangle = \frac{B}{2}\Gamma\left(\frac{1}{2} - \frac{z}{B}\right)U\left(\frac{1}{2} - \frac{z}{B}, 1, \frac{B}{2}|\mathbf{r} - \mathbf{r}'|^2\right) \times \exp\left(-\frac{B}{4}|\mathbf{r} - \mathbf{r}'|^2 + \frac{\mathrm{i}B}{4}(x + x')(y - y')\right). \tag{A.4}$$

The presence of the Euler Γ function indicates that the Landau levels remain unchanged on the cylinder, and U is the Kummer function [16]. By using some technical estimates as in [17] one may show that for $\frac{B}{2} < \mathrm{Re}\, z < \frac{3B}{2}$ the absolute value of (A.3) is bounded above by the simple expression

$$\frac{cB}{\delta_0(z)}\,\mathrm{e}^{-\frac{B}{8}|x-x'|^2}\sum_{m=-1,0,+1} S(x - x', y - y' - mL)\,\mathrm{e}^{-\frac{B}{8}(y-y'-mL)^2} \tag{A.5}$$

where c is a numerical constant independent of B and L. The factor S comes from the logarithmic divergence at coincident points

$$\begin{aligned} S(x - x', y - y') &= 1 && \text{for } \frac{B}{2}|\mathbf{r} - \mathbf{r}'|^2 > 1 \\ &= \ln\frac{B}{2}|\mathbf{r} - \mathbf{r}'|^2 && \text{otherwise.} \end{aligned} \tag{A.6}$$

A bound similar to (A.5) holds for $|\partial_x\langle \mathbf{r}|G_{0,\infty}(z)|\mathbf{r}'\rangle|$, with cB replaced by $cB^{\frac{3}{2}}$ and S replaced by $\frac{|x-x'|}{|\mathbf{r}-\mathbf{r}'|^2}$ when $\frac{B}{2}|\mathbf{r} - \mathbf{r}'|^2 < 1$. The important feature for the subsequent estimates is that all the above singularities are integrable. In what follows c denotes a generic numerical positive constant.

A.1. Estimate of $\|K_e\|$

From the resolvent identity

$$K_e(z) = \tfrac{1}{2}\left[p_x^2, J_e\right]G_0(z)\chi_e + \tfrac{1}{2}\left[p_x^2, J_e\right]G_0(z)WG_e(z)\chi_e. \tag{A.7}$$

Evaluating the commutator, and using $\|G_e(z)\| \leqslant \delta_e(z)^{-1}$ we find

$$\|K_e(z)\| \leqslant \tfrac{1}{2}\|J_e''G_0(z)\chi_e\| + \|J_e'\partial_x G_0(z)\chi_e\| + \delta_e(z)^{-1}(\|J_e''G_0(z)W\| + \|J_e'\partial_x G_0(z)W\|). \tag{A.8}$$

Estimate (2.7) follows from the fact that all norms on the right-hand side of (A.8) involve matrix elements of $G_0(z)$, and $\partial_x G_0(z)$ separated by a distance at least equal to $\frac{D}{4}$. We use the estimate (A an operator with kernel $A(\mathbf{r}, \mathbf{r}')$)

$$\|A\| \leqslant \max\left(\sup_{\mathbf{r}'}\int \mathrm{d}\mathbf{r}|A(\mathbf{r}, \mathbf{r}')|; \sup_{\mathbf{r}}\int \mathrm{d}\mathbf{r}'|A(\mathbf{r}, \mathbf{r}')|\right). \tag{A.9}$$

For the first norm we have

$$\int_{-\frac{3D}{4}-1}^{-\frac{3D}{4}+1} \mathrm{d}x \int_{\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y\, J_e''(x)|\langle \mathbf{r}|G_0(z)|\mathbf{r}'\rangle|\chi_e(x') \leqslant \frac{cBL}{\delta_0(z)} \int_{-\frac{3D}{4}-1}^{-\frac{3D}{4}+1} \mathrm{d}x\, \mathrm{e}^{-\frac{B}{8}|x-x'|^2}\chi_e(x')$$
$$\leqslant \frac{c\sqrt{B}L}{\delta_0(z)}\,\mathrm{e}^{-cBD^2}. \tag{A.10}$$

In the first inequality we used (A.5) and in the last one we use the fact that $|x-x'| \geqslant \frac{D}{4}$. On the other hand

$$J_e''(x)\int_{-\frac{D}{2}}^{\infty} \mathrm{d}x' \int_{-\frac{L}{2}}^{\frac{L}{2}} \mathrm{d}y'|\langle \mathbf{r}|G_0(z)|\mathbf{r}'\rangle|\chi_e(x') \leqslant \frac{cBL}{\delta_0(z)} J_e''(x)\int_{-\frac{D}{2}}^{\infty} \mathrm{d}x'\, \mathrm{e}^{-\frac{B}{8}|x-x'|^2}\chi_e(x')$$
$$\leqslant \frac{c\sqrt{B}L}{\delta_0(z)}\,\mathrm{e}^{-cBD^2}. \tag{A.11}$$

Thus $\|J_e''G_0(z)\chi_e\| \leqslant \frac{CL^2}{\delta_0(z)}\,\mathrm{e}^{-cBD^2}$. For the term involving $\partial_x G_0(z)$ the estimates are similar. The terms involving W lead to the same estimates provided

$$\int_{-\frac{3D}{4}-1}^{-\frac{3D}{4}+1} \mathrm{d}x\, \mathrm{e}^{-\frac{B}{8}|x-x'|^2}U(x') \qquad \text{and} \qquad J_e''(x)\int_0^{\infty} \mathrm{d}x'\, \mathrm{e}^{-\frac{B}{8}|x-x'|^2}U(x') \tag{A.12}$$

are bounded by $O(\exp(-cBD^2))$. This is the case for the class of functions $W(x)$ that grow polynomially as $x \to +\infty$.

A.2. Estimate for $\|K_b\|$

First we sketch the derivation of an estimate for the kernel of $G_b(z)$ and its derivative for z in the gap of $\sigma(H_b(\Phi))$.

$$\langle \mathbf{r}|G_b(z)|\mathbf{r}'\rangle = \langle \mathbf{r}|G_0(z)|\mathbf{r}'\rangle + \sum_{m\geqslant 1}\int \mathrm{d}\mathbf{r}_1 \ldots \int \mathrm{d}\mathbf{r}_m \langle \mathbf{r}|G_0(z)|\mathbf{r}_1\rangle V(\mathbf{r}_1)$$
$$\times \langle \mathbf{r}_1|G_0(z)|\mathbf{r}_2\rangle V(\mathbf{r}_2)\ldots V(\mathbf{r}_m)\langle \mathbf{r}_m|G_0(z)|\mathbf{r}'\rangle. \tag{A.13}$$

Here the range of the integrals over $x_1, \ldots, x_m$ is $]-\infty, +\infty[$, and that of $y_1, \ldots, y_m$ is $\left[-\frac{L}{2}, \frac{L}{2}\right]$. In order to extract the decay for $|x-x'|$ large from (A.13) and (A.5) we use, from $B|x-x'|^2 > 2\sqrt{B}|x-x'| - 1$,

$$\mathrm{e}^{-\frac{B}{8}(|x-x_1|^2+|x_1-x_2|^2+\cdots+|x_m-x'|^2)} \leqslant \mathrm{e}^{-\frac{B}{16}(|x-x_1|^2+|x_1-x_2|^2+\cdots+|x_m-x'|^2)}\mathrm{e}^{-\frac{\sqrt{B}}{8}(|x-x_1|+|x_1-x_2|+\cdots+|x_m-x'|)}\,\mathrm{e}^{\frac{m}{16}}$$
$$\leqslant \mathrm{e}^{\frac{m}{16}}\,\mathrm{e}^{-\frac{\sqrt{B}}{8}|x-x'|}\,\mathrm{e}^{-\frac{B}{16}(|x-x_1|^2+|x_1-x_2|^2+\cdots+|x_m-x'|^2)}. \tag{A.14}$$

Thanks to (A.5), (A.13), (A.14) we obtain for $\frac{B}{2}|x-x'| > 1$

$$|\langle \mathbf{r}|G_b(z)|\mathbf{r}'\rangle| \leqslant \frac{cB}{\delta_0(z)}\,\mathrm{e}^{-\frac{B}{8}|x-x'|^2} + \sum_{m\geqslant 1}\left(\frac{cB}{\delta_0(z)}\right)^{m+1}\left(\frac{w}{B}\right)^m\,\mathrm{e}^{-\frac{\sqrt{B}}{8}|x-x'|}$$
$$\leqslant \frac{cB}{\delta_0(z)-cw}\,\mathrm{e}^{-\frac{\sqrt{B}}{8}|x-x'|}. \tag{A.15}$$

This bound is valid as long as w is small enough. Clearly from (A.13), following the same steps, we obtain a similar inequality, with cB replaced by $cB^{\frac{3}{2}}$, for $|\partial_x\langle \mathbf{r}|G_b(z)|\mathbf{r}'\rangle|$ if $\frac{B}{2}|x-x'| > 1$.

To estimate $\|K_b\|$ we have to compute the norms on the right-hand side of

$$\|K_b\| \leqslant \tfrac{1}{2}\|J_b''G_b(z)\chi_b\| + \|J_b'\partial_x G_b(z)\chi_b\|. \tag{A.16}$$

This can be done easily using (A.9), (A.16) and the bound (A.15) together with that on the derivative. Then one finds

$$\|K_b\| \leqslant \frac{cB^{\frac{3}{2}}L}{\delta_0(z) - cw}\,\mathrm{e}^{-c\sqrt{B}D}. \tag{A.17}$$

Appendix B

By Cauchy's formula, and the resolvent identity

$$P_{F,0}^D = \int_{\Gamma_F} \mathrm{d}z \frac{1}{z - H_D(0)} = \int_{\Gamma_F} \mathrm{d}z \frac{1}{z - H_0(0)} + \int_{\Gamma_F} \mathrm{d}z \frac{1}{z - H_D(0)} (W + V_D) \frac{1}{z - H_0(0)} \tag{B.1}$$

where the contour Γ_F encloses the part of the spectrum of $H_D(0)$ lying below E_F. Setting $g = UU_{T-s}^D(V - V_D)$ we have for the Hilbert–Schmidt norm

$$\begin{aligned}\|P_{F,0}^D g\|_{HS} \leqslant |\Gamma_F| \sup_{z\in\Gamma_F} \left\| \frac{1}{z - H_0(0)} g \right\|_{HS} + \frac{|\Gamma_F|}{\operatorname{dist}(E_F, \sigma(H_D(0)))} \\ \times \left(\sup_{z\in\Gamma_F} \left\| W \frac{1}{z - H_0(0)} g \right\|_{HS} + w \sup_{z\in\Gamma_F} \left\| \frac{1}{z - H_0(0)} g \right\|_{HS} \right).\end{aligned} \tag{B.2}$$

Here $|\Gamma_F|$ is the length of the contour which is finite because the spectrum is bounded below. Since $V - V_D$ has compact support, g is a square integrable function on the cylinder. Therefore, from the bound (A.5), (A.6) on the kernel of $(z - H_0)^{-1}$ it is easily seen that all the Hilbert–Schmidt norms in (B.2) are finite. These norms can be bounded above uniformly in $0 \leqslant s \leqslant T$, and the supremum over z stays finite as long as the contour does not touch a Landau level. Therefore (3.11) is finite.

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